Design Sensitivity Analysis of Boundary-Element Substructures

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A design sensitivity analysis (DSA) formulation using substructuring is presented for the boundary-element method (BEM). A reduced set of DSA equations is obtained from the implicit differentiation of reduced BEM equations. Relationships for the expansion of the reduced DSA expressions to determine the sensitivities of all condensed quantities that are present in the analysis are also given. These reduced and expansion sensitivity equations involve the sensitivity of the inverse of a system submatrix. A procedure is developed to obviate the need for the determination of this submatrix inverse sensitivity. The present formulation allows for arbitrary condensing and noncondensing of zones in multiple zone models, and such zones can simultaneously exist in the reduced set of analysis equations. A set of numerical examples for two-dimensional plane and axisymmetric continua are presented. Since the condensation formulation developed is exact, the accuracies of these examples are the same as those for the uncondensed model. It is shown, however, from the CPU timings presented for these examples, that the substructuring technique can dramatically economize the numerical shape optimization of models with partially sensitive geometries.

Introduction

THE boundary-element method (BEM) presents many advantages for its use in shape-optimization applications. These advantages were recently outlined in Ref. 1. Much of the literature concerning sensitivity formulations for structural continua using BEM has appeared in just the past three years. A finite-difference approach for shape optimization was given by Wu.[2] Design sensitivity formulations were given for plane two-dimensional continua by Barone and Yang[3] and by Kane and Saigal,[4] for axisymmetric continua by Saigal et al.[5] and by Rice and Mukherjee,[6] and for three-dimensional continua by Aithal et al.[7] A boundary-element formulation for design sensitivities in materially nonlinear problems has recently been developed by Mukherjee and Chandra.[8] A survey of other sensitivity formulations in BEM for structural sizing, heat-transfer, aerodynamic, and other applications may be found in, among others, Ref. 9.

Apart from the numerous advantages offered by BEM, both in structural analysis and in sensitivity analysis, it results in a fully populated system of matrix equations which are generally uneconomical to solve. The subdivision of the continua into zones helps in obtaining matrix equations with a blocked sparse structure.[10] Further economy in the solution of BEM equations may be obtained through the concept of static condensation of degrees of freedom. The efficiency of such a procedure for boundary-element analysis was demonstrated by Kane and Saigal[11] where they have also presented a survey of other efforts concerning condensation in BEM.

The process of numerical shape optimization involves beginning with an initial design, and generating a sequence of improving designs in an iterative algorithm to obtain the final optimal shape. A reduction in the size of the model being iterated upon in this process would clearly increase the ability to treat presently untractable large systems. The present study is directed towards such a goal by reducing the model to retain only the degrees of freedom representing the parts of the design that are allowed to change during the optimization loop. The multizone BEM is especially suited for condensation since the zones can be defined such that the degrees of freedom to be retained appear in separate zones. The rest of the zones can then be condensed to obtain a reduced system of equations.

In the present paper, a reduced set of BEM equations is first written for the analysis of the response of the structural system. These equations are written for multiple boundary-element zones and allow for the simultaneous presence of multiple condensed and multiple uncondensed zones.[11] A relationship for the expansion of the solution to obtain the unknown condensed quantities is also given. Implicit differentiation of the reduced BEM equations is performed to obtain a set of reduced design sensitivity analysis (DSA) equations. The expression for unknown condensed sensitivities is also developed through the use of implicit differentiation of the expansion equations for the analysis. Both the reduced sensitivity equations and the sensitivity equations obtained from the substructure expansion formulation appear to involve the computation of the sensitivity of the inverse of a submatrix (partition) of the system matrix. A method is presented to alleviate the need for the computation of the sensitivity of this submatrix inverse.

Sandgren and Wu[12] have recently presented a description of shape optimization of objects using BEM with substructuring. The economies associated with condensing the geometrically insensitive boundary-element zones are pointed out by them. In their procedure, however, the sensitivities of response quantities are obtained by a sequence of univariate changes in each design variable, followed by complete separate reanalysis for each of the perturbed models generated. The incorporation of the responses of these perturbed models in the finite-difference approximations provides the required design sensitivities. An analytical approach for sensitivity computations is necessary because the alternative finite-difference approach depends for its accuracy on the step size chosen for the finite-difference approximation.[13] For large complex systems, the

Received Dec. 18, 1988; revision received July 4, 1989.

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choice of the step size can be cumbersome and may lead to erroneous results. The present formulation develops the analytical sensitivity expressions and thus avoids the above-mentioned problems.

A number of examples using the substructured sensitivity formulation were studied. The CPU timings for both the full model and the substructured models were recorded. The substructured sensitivity analysis requires slightly more CPU time for the initial setup of the condensed matrices, their storage, and assembly. However, the solution step, which is repeated numerous times in a numerical shape-optimization procedure, shows a saving in CPU time of approximately 50% in each solution pass for the substructured case compared to the uncondensed case. A similar saving is seen also in the matrix factorization step which is performed only once. The cumulative savings for substructured sensitivity analysis are shown to be substantial, thus demonstrating that the present formulations lead to a powerful tool for economical DSA of large continua.

Design Sensitivity Analysis with Substructures

Boundary-Element Equations with Substructures

The theoretical foundation of boundary-element analysis based on Somigliana's identity for elastostatics is available in text books; for example, by Banerjee and Butterfield.[11] The advantages of dividing the continuum into multiple zones for boundary-element modeling in increasing the efficiency and accuracy of the solution are well known. The assembled boundary-element equations can be written in the matrix form as

$$[F]\{U\} = [G]\{t\} + \{f\} \tag{1}$$

$[F]$ and $[G]$ are the boundary-element system matrices; $\{u\}$ and $\{t\}$ are the vectors of nodal displacements and tractions, respectively; and $\{f\}$ is a vector including the effects of body forces, such as thermal, gravitational, centrifugal, etc. The system matrices in Eq. (1) are sparse blocked for multizone models and are fully populated for single zone models. Representative matrix populations for such models were shown in Ref. 11.

Equation (1) can be partitioned as

$$\begin{bmatrix} [F_{MM}] & [F_{MC}] \\ [F_{CM}] & [F_{CC}] \end{bmatrix} \begin{Bmatrix} \{u_M\} \\ \{u_C\} \end{Bmatrix} = \begin{bmatrix} [G_{MM}] & [G_{MC}] \\ [G_{CM}] & [G_{CC}] \end{bmatrix} \begin{Bmatrix} \{t_M\} \\ \{t_C\} \end{Bmatrix} + \begin{Bmatrix} \{f_M\} \\ \{f_C\} \end{Bmatrix} \tag{2}$$

where the subscripts M and C refer to the master and condensed degrees of freedom (DOF), respectively. Equation (2) can be rearranged as shown in Ref. 11 to obtain the condensed boundary-element zone matrix equation in terms of the unknown master DOF's $\{u_M\}$ as

$$[M_1]\{u_M\} = [M_2]\{t_M\} + [M_3]\{t_c\} + [M_4]\{f_C\} + \{f_M\} \tag{3}$$

where

$$[M_1] = [F_{MM}] - [F_{MC}][F_{CC}]^{-1}[F_{CM}] \tag{4}$$

$$[M_2] = [G_{MM}] - [F_{MC}][F_{CC}]^{-1}[G_{CM}] \tag{5}$$

$$[M_3] = [G_{MC}] - [F_{MC}][F_{CC}]^{-1}[G_{CC}] \tag{6}$$

$$[M_4] = -[F_{MC}][F_{CC}]^{-1} \tag{7}$$

The solution of the system can then be obtained by solving the reduced system of Eqs. (3). The remaining unknowns $\{u_C\}$ may be determined from the boundary-element zone matrix expansion equation given as

$$\{u_C\} = [F_{CC}]^{-1}[G_{CM}]\{t_M\} + [F_{CC}]^{-1}[G_{CC}]\{t_C\} - [F_{CC}]^{-1}[F_{CM}]\{u_M\} + [F_{CC}]^{-1}\{f_C\} \tag{8}$$

An efficient implementation and the details of substructured boundary-element analysis are given in Ref. 11. It was shown that Eqs. (3-7) can be used to produce condensed contributions at the individual BEM zone level. These condensed contributions can then be directly used for assemblage into the overall multizone BEM system equations.

Implicit Differentiation of Substructured Boundary-Element Equations

The implicit-differentiation approach for design sensitivity analysis with boundary elements involves the differentiation of the boundary-element equations [Eq. (1)] with respect to each design variable.[4] In the present study, the implicit differentiation of the condensed system of Eqs. (3) is performed. This leads to a system of condensed sensitivity equations involving the sensitivities of the unknown quantities corresponding to the master degrees of freedom only as

$$[M_1]\{u_M\}_{,L} = [M_2]\{t_M\}_{,L} + \{b_1\} \tag{9}$$

where

$$\{b_1\} = -[M_1]_{,L}\{u_M\} + [M_2]_{,L}\{t_M\} + [M_3]_{,L}\{t_C\} + [M_3]\{t_C\}_{,L} + [M_4]_{,L}\{f_C\} + [M_4]\{f_C\}_{,L} + \{f_M\}_{,L} \tag{10}$$

The key step in the determination of sensitivities of surface response quantities at the master nodes in this approach is the computation of sensitivities of the $[M_i]$ matrices. These are evaluated as

$$[M_1]_{,L} = [F_{MM}]_{,L} - \{[F_{MC}]_{,L}([F_{CC}]^{-1}[F_{CM}]) + [F_{MC}]([F_{CC}]^{-1}[F_{CM}])_{,L}\} \tag{11}$$

$$[M_2]_{,L} = [G_{MM}]_{,L} - \{[F_{MC}]_{,L}([F_{CC}]^{-1}[G_{CM}]) + [F_{MC}]([F_{CC}]^{-1}[G_{CM}])_{,L}\} \tag{12}$$

$$[M_3]_{,L} = [G_{MC}]_{,L} - \{[F_{MC}]_{,L}([F_{CC}]^{-1}[G_{CC}]) + [F_{MC}]([F_{CC}]^{-1}[G_{CC}])_{,L}\} \tag{13}$$

and

$$[M_4]_{,L} = -([F_{MC}][F_{CC}]^{-1})_{,L} \tag{14}$$

It is noted that $\{t_C\}_{,L}$ in Eq. (10) is generally equal to zero since $\{t_C\}$ are the tractions over the unchanging portion of the body during the shape-optimization process. It is also possible to obtain displacement and traction component sensitivities for degrees of freedom that have been condensed. This formulation for recovering sensitivities of condensed quantities can be derived by implicit differentiation of the substructure expansion Eqs. (8).

$$\{u_C\}_{,L} = ([F_{CC}]^{-1}[G_{CM}])_{,L}\{t_M\} + ([F_{CC}]^{-1}[G_{CM}])\{t_M\}_{,L} + ([F_{CC}]^{-1}[G_{CC}])_{,L}\{t_C\} - ([F_{CC}]^{-1}[F_{CM}])_{,L}\{u_M\} - ([F_{CC}]^{-1}[F_{CM}])\{u_M\}_{,L} + ([F_{CC}]^{-1}\{f_C\} \tag{15}$$

It is thus possible to obtain complete sensitivity information for the boundary-element substructures using Eqs. (9-15) provided all of the matrix entries present in these equations can be computed. The computation of the submatrices $[F_{ij}]$, $i,j = M,C$, has been discussed in earlier papers.[4,5,7]

Sensitivity of Inverse of Matrix $[F_{CC}]$

As indicated by Eqs. (11-14), the computation of the matrix sensitivities $[M_i]_{,L}$, $i = 1,2,3,4$, requires the computation of the sensitivity of the matrix $[F_{CC}]^{-1}$. This apparent requirement has inhibited the authors for awhile from using substructuring for design sensitivity analysis. A closer look reveals that the matrix $[F_{CC}]^{-1}$ appearing in the DSA formulation is always postmultiplied or premultiplied by another matrix. This fact is used to obviate the explicit determination of $([F_{CC}]^{-1})_{,L}$ as follows.

Consider, for example, the matrix product appearing in Eq. (11):

$$[F_{CC}]^{-1}[F_{CM}] = [D] \tag{16}$$

The sensitivity $[D]_{,L}$ is then required for the computation of the matrix $[M_1]_{,L}$ in Eq. (11). Equation (16) can be rewritten as

$$[F_{CM}] = [F_{CC}][D] \tag{17}$$

The implicit differentiation of Eq. (17) leads to

$$[F_{CM}]_{,L} = [F_{CC}]_{,L}[D] + [F_{CC}][D]_{,L} \tag{18}$$

or, on rearranging,

$$[F_{CC}][D]_{,L} = [F_{CM}]_{,L} - [F_{CC}]_{,L}[D] \tag{19}$$

The required sensitivity $[D]_{,L}$ can then be determined by solving Eq. (19). The matrix term $[D]_{,L}$ is obtained by forward reduction and backward substitution of the columns of the right-hand side matrix shown in Eq. (19). This is very efficient since the triangular factors of the matrix block $[F_{CC}]$ computed during the condensation step for evaluating Eqs. (4-7) of the analysis can be saved and reused for determining $[D]_{,L}$. The sensitivities of other matrix products involving $[F_{CC}]^{-1}$ and given in Eqs. (12-14) can be determined in an analogous fashion. The sensitivities of matrix $[F_{CC}]^{-1}$ also appear in the expansion Eq. (15) for sensitivities and is again treated in a similar fashion. The block entries shown in the condensed sensitivity Eq. (9) can thus all be obtained. The procedure for the solution of the condensed equations [Eq. (9)] presented here is the same as for the uncondensed equations and is described in detail in Ref. 4. The present substructured formulation retains the major advantage of the implicit-differentiation approach for DSA of the ability to reuse the triangular factors of the boundary-element system coefficient matrix which are formed and saved during the analysis step. An additional advantage of this formulation is due to the fact that the multiple block forward reductions and back substitutions associated with multiple design variables, X_L $(L = 1,2...)$, are performed using triangular factors and right-hand side vectors of reduced sizes resulting in a considerable economy in the overall solution process.

Numerical Results

A series of numerical examples are presented to demonstrate the accuracy and efficiency of the DSA formulation with substructures as described above. The advantage of substructuring for objects with geometric sensitivity confined to only a portion of the object is shown through numerical data. Since the condensation and subsequent expansion steps are exact, the accuracy of DSA results obtained using substructuring is the same as that for DSA results without substructuring. The main steps for the boundary-element response analysis phase with substructuring include: accounting for the array storage required for the sparse blocked matrix procedures, integration of the kernels to obtain entries in these matrix equations, first assembly of system matrices of the individual zone, optional condensation of specified zones, second assembly of zone contributions into the overall multizone sparse blocked system matrices with both condensed and uncondensed zones, sparse blocked matrix factorization, solution of equations and optional expansion of solution to obtain detailed results for any condensed zone, and stress recovery. Additional steps for design sensitivity analysis include: additional accounting for the storage of the matrix blocks for sensitivity analysis, integration of sensitivity kernels to obtain sensitivity matrix block entries, first assembly of sensitivity system matrices, optional condensation, second assembly with both condensed and uncondensed matrices, solution for sensitivities and recovery of solution for condensed zones, and stress sensitivity recovery. It is noted that the matrix factorization step is required only once during the response analysis phase. Since this matrix appears as the coefficient matrix for the sensitivity analysis phase also, the same factorization can be reused for each sensitivity calculation.

Another important strategy used in the DSA is sparsity exploitation. During each step of the DSA procedure described above, the null or unchanging quantities can be detected in advance and the operations for calculation of these quantities can be skipped. For example, the block entries for sensitivities in boundary-element zones that are geometrically insensitive to design variable X_L are null entries, and need never be computed or condensed. Partial geometric insensitivity within boundary-element zones can also be similarly monitored and exploited to avoid superfluous operations on individual null rows and columns in the resulting matrix sensitivity blocks. This strategy allows for substantial economies to be realized in an overall shape-optimization system.

In a complete shape-optimization process, the sensitivities usually need to be computed many times since the optimal shape is evolved in an iterative fashion. It is thus important that the CPU time for each solution step for both the response analysis phase and the sensitivity analysis be kept to a minimum. For substructured design sensitivity analysis, the initial accounting and setup of the problem usually consumes slightly more CPU time than the full sensitivity analysis. This additional time is, however, easily offset by the savings obtained due to the solving of a considerably smaller system of equations for each reduced analysis and sensitivity calculation step during the shape-optimization process. For the examples shown below, the CPU times for each step mentioned above are given. For each case the reduced solution step, subsequent to the first analysis step, is seen to consume a considerably lesser CPU time. This clearly demonstrates, then, the advantages of substructuring for design sensitivity analysis using boundary elements.

All of the computations, for results and CPU timings, were carried out on a RIDGE 3200 computer system. Small variations in CPU timings for the same problem were observed depending on whether the problems were executed during the "busy" or the "quiet" periods for the computer system.

Fillet Attachment Subjected to an Axial Pull

The filleted attachment for a rod shown in Fig. 1 was subjected to an axial pull by applying a traction of 1000 psi to its right end. The fillet problem has commonly been investigated for design sensitivity and shape-optimization calculations. The fillet model was divided into three boundary-element zones. Zones I, II, and III were modeled using 32, 34, and 20 quadratic continuous boundary elements, respectively. In this design of the fillet, an improved shape was desired for the curved portion of the fillet in boundary-element zone II, keeping the rest of the geometry the same. The weight of zone II was chosen as the objective function. Also, the von Mises stress along the curved portion of zone II was constrained below a certain value lower than that obtained from the initial design. Thus, the design sensitivities of zone II only need to be computed. In the present investigation, the design sensitivities

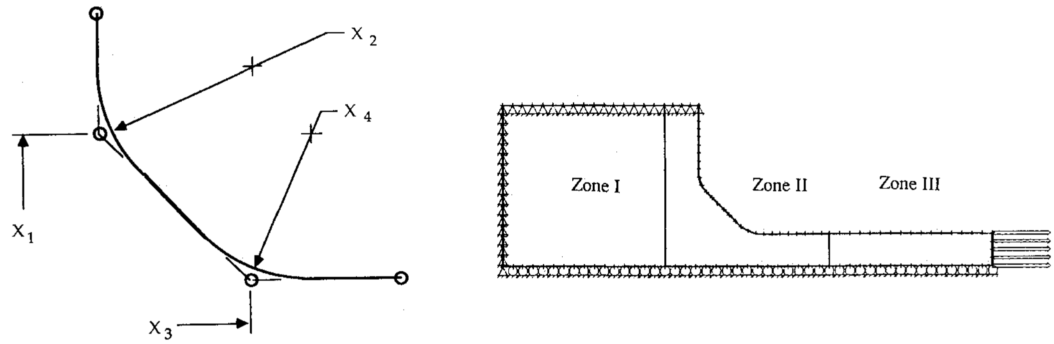

Fig. 1 Filleted attachment shape-optimization example. Design variables, three-zone model, and boundary conditions.

were first obtained using the entire, uncondensed model consisting of the three boundary-element zones. Zones I and III were next condensed to obtain the same design sensitivities again. The CPU times and accuracies for both these cases were obtained for comparison.

The four design variables chosen to control the shape of the fillet are shown in Fig. 1. The timings for the steps involved in both the response analysis and the design sensitivity analysis are shown in Table 1. The sensitivity analysis timings shown are for one design variable only. A saving of 60% of CPU time is seen for the matrix factorization step for the substructured model. Also of significance are the savings in CPU time observed in item 14 of table for the solution of the sensitivity equations. For multiple design variables, and for the multiple calculations of design sensitivities for these design variables, this 45% saving in CPU time per calculation represents an significant saving in CPU time and clearly offsets any extra time that was needed for the setup of the substructured analysis, such as in items 4 and 12 of the table. The advantages of the new substructuring formulation presented in this study are thus evident.

The filleted attachment problem was then used in a shape-optimization procedure where 48 response analyses and six design sensitivity analyses for four design variables were performed. Selected evolving shapes and von Mises stress responses for this shape optimization have been shown in Fig. 2. In Table 1, the computer resources required in the shape optimization are tabulated. In analyses that occur after the first analysis, the shape-optimization system monitors the fact that zones I and III are geometrically insensitive to the design variables. The system exploits this fact, and avoids most recomputations of unchanging data in the analyses subsequent to the first analysis. This is why some of the items 1A–8A in Table 1 are smaller than the corresponding items 1–8 in the same table. From this data, it is demonstrated that substructuring considerably reduces the computer resources required

Table 1 CPU timings for substructured and nonsubstructured analysis, DSA, and shape optimization for the filleted attachment example

Item no.	Step description	CPU times: UP/FD[a]	No condensation	Zones I & II condensed
	First response analysis			
1	Accounting	2.0	2.0	2.0
2	Numerical integration	15.5	15.5	15.5
3	First assembly (zone level)	5.8	5.8	5.8
4	Condensation	——	——	8.6
5	Second assembly (overall level)	0.3	0.3	0.1
6	Matrix factorization	19.0	19.0	7.7
7	Solution (fwd. reduce, bk. substitute)	1.3	1.3	0.6
8	Surface stress recovery	2.6	2.6	2.6
	All subsequent response analysis			
1A	Accounting	2.0	——	——
2A	Numerical integration	15.5	5.2	5.2
3A	First assembly (zone level)	5.8	1.9	1.9
4A	Condensation	——	——	——
5A	Second assembly (overall level)	0.3	0.1	0.1
6A	Matrix factorization	19.0	19.0	7.7
7A	Solution (fwd. reduce, bk. substitute)	1.3	1.3	0.6
8A	Surface stress recovery	2.6	2.6	2.6
	Design sensitivity analysis (one design variable)			
9	Accounting	2.0	——	——
10	Numerical integration	15.5	5.6	5.6
11	First assembly (zone level)	5.8	1.9	1.9
12	Condensation	——	——	——
13	Second assembly (overall level)	0.3	0.2	0.1
13A	Matrix factorization	19.0	——	——
14	Solution (fwd. reduce, bk. substitute)	1.3	1.3	0.6
15	Surface stress sensitivity recovery	2.6	1.1	1.1
	Shape optimization (48 analyses, 6 DSA's with 4 design variables)			
16	First response analysis	46.5	46.5	42.9
17	47 Subsequent response analyses	$47 \times 46.5 = 2185.5$	$47 \times 30.1 = 1414.7$	$47 \times 18.1 = 850.7$
18	6 Design sensitivity analyses	$24 \times 46.5 = 1116.0$	$24 \times 10.1 = 242.4$	$24 \times 9.3 = 223.2$
19	Total shape-optimization time	3348.0	1703.6	1116.8

[a]UP/FD: univariate perturbation/forward difference.

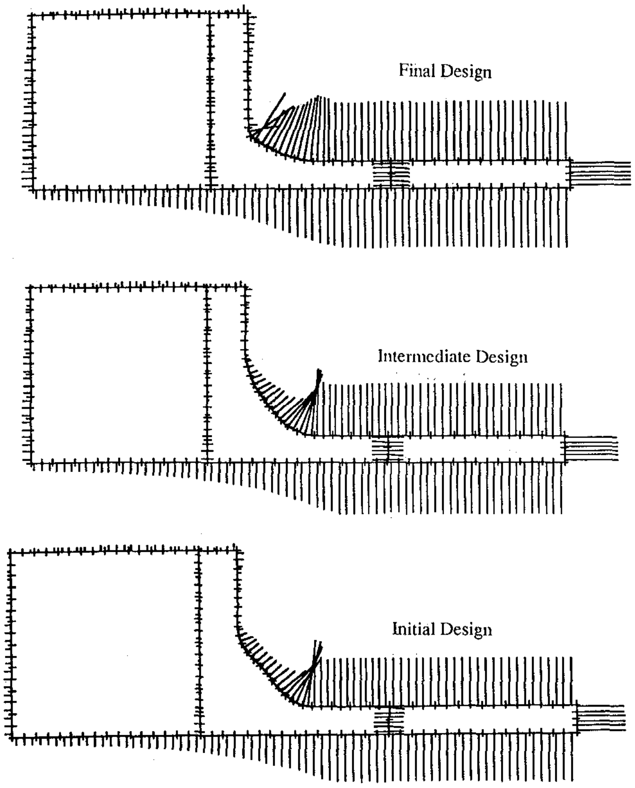

Fig. 2 Filleted attachment example. Evolving shapes and responses.

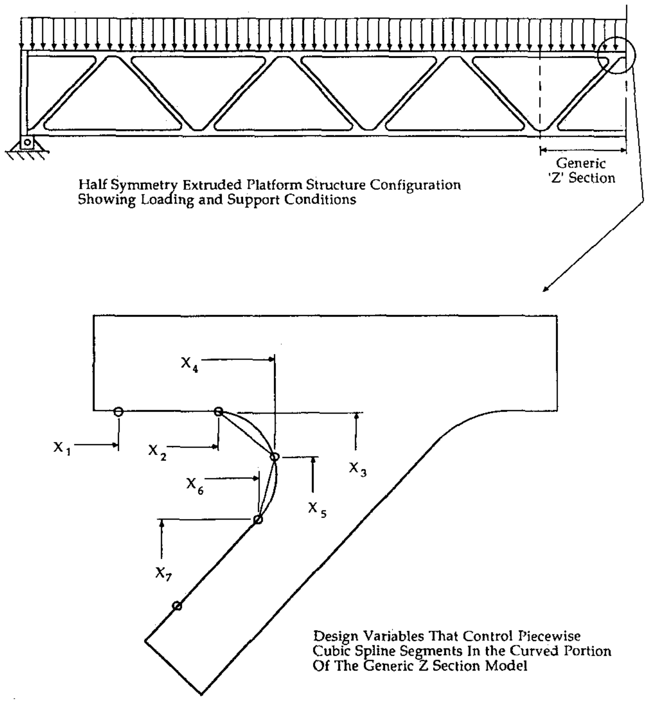

Fig. 3 Extruded platform shape-optimization example. Overall configuration and design variables that control the variable curved portion of the design.

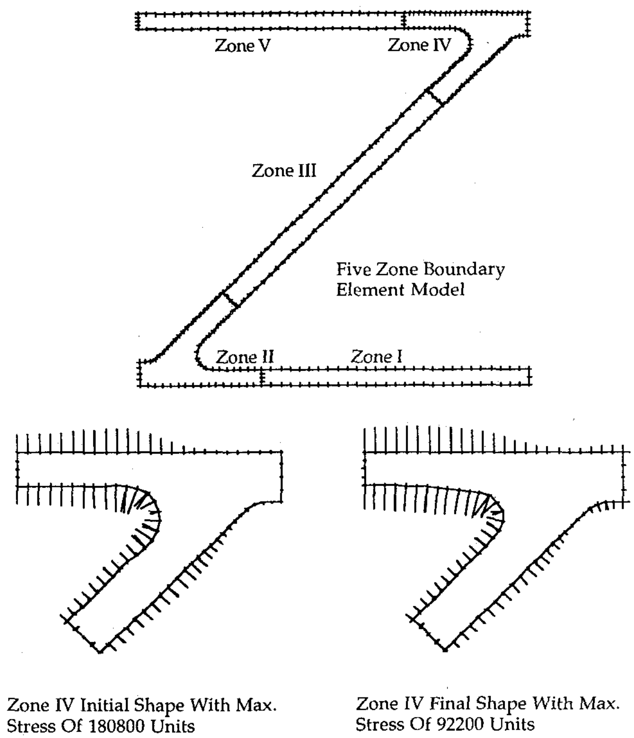

Fig. 4 Extruded platform example. "Z"-shape model and responses.

to perform shape optimization. In the first column of this table, the timings associated with the univariate perturbation approach to shape optimization[13] are also shown for comparison.

Extruded Platform Configuration Subjected to Bending

Extruded structural members are frequently employed in aerospace and civil engineering structures. As an example of the usefulness of the condensed DSA technique in the shape optimization of such designs, the platform structure shown in Fig. 3 has been chosen for investigation. The generic "Z" section that represents the repeated geometry of this extruded shape is also shown in this figure. This generic section was modeled using the five-zone boundary-element model shown in Fig. 4, with appropriate loads and boundary conditions used to simulate the rest of the structure. These five zones have been modeled using 31, 34, 36, 44, and 34 quadratic boundary elements, respectively. The seven design variables for this case were shown in Fig. 3, and it is seen that they affect only the geometry of the curved portion of zone IV. Geometries of zones I, II, III, and V are not affected by a change in these design variables. Since these zones do not exhibit any geometric sensitivity with respect to the design variables, they may be condensed to gain economy in the analysis and design sensitivity calculations. CPU timings were obtained for this case and also for the case in which no substructuring is done to compare the extent of the savings resulting from the present condensation formulation.

The CPU timings for both the full analysis and the substructured analysis were shown in Table 2. To provide an insight into the reasons for the difference in the timings shown in Table 2 for the full and substructured models used in the shape-optimization process, the left-hand side sparse blocked matrix populations are shown in Fig. 5 for both these cases. The matrices for both the full model and the substructured model have been drawn to an identical scale. The difference in the sizes of these two matrices indicates the substantial benefit associated with condensing unchanging zones in problems with such evolving shapes. For example, a 63% savings is obtained in the factorization step as seen from item 6 of Table 2. Furthermore, the solution step in design sensitivity analysis, which is repeated numerous times during shape optimization, depicts a saving in CPU time of approximately 55% for each design variable per design sensitivity analysis. These savings are seen in item 14 of Table 2. Again, these savings easily offset the additional computation burden incurred due to the condensation subroutines and the setup for condensation formulation. The accuracies of the von Mises stress sensitivities obtained using the full analysis and the substructured analysis

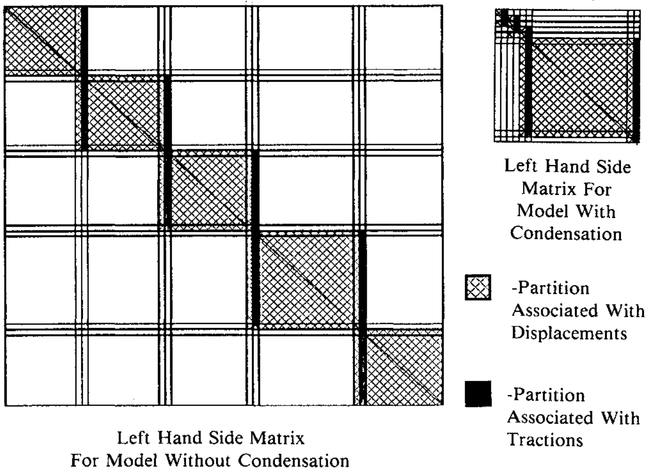

Fig. 5 Comparison of left-hand-side matrices for the five-zone "Z" model with and without condensation.

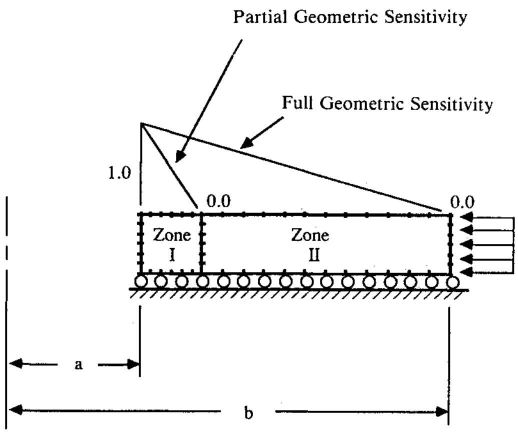

Fig. 6 Thin circular disk. Axisymmetric model used to examine the effect of partial geometric sensitivity on the efficiency of the substructured design sensitivity analysis.

were compared. Because of the exact nature of the condensation procedure, these results agree up to five significant decimal places. The initial and final shapes of the Z-configuration as they evolve to an optimum shape are shown in Fig. 4. The timings for the univariate perturbation approach are also shown in column 1 of Table 2 to show the extent of savings obtained from the use of the present approach.

Thin Circular Disk Under External Pressure

The computations for displacement and stress sensitivities of a thin circular disk under external pressure shown in Fig. 6 were studied. Axisymmetric boundary elements were used to model a radial cross section of the circular disk. The cross section was divided into two equal zones and a total of 30 boundary elements were used. Two example models of the disk, one with full radial geometric sensitivity and the other with partial radial geometric sensitivity, were considered. These sensitivities of the boundary-element nodes can be written as shown below.

Full radial sensitivity:

$$R = a + t(b - a) \qquad 0 \leq t \geq 1$$
$$R_{,a} = 1 - t \qquad 0 \leq t \geq 1$$

Table 2 CPU timings for substructured and nonsubstructured analysis, DSA, and shape optimization for the extruded platform example

Item no.	Step description	CPU times: UP/FD[a]	No condensation	Zones I, II, III, & V condensed
	First response analysis			
1	Accounting	6.0	6.0	6.0
2	Numerical integration	39.0	39.0	39.0
3	First assembly (zone level)	14.4	14.4	14.4
4	Condensation	——	——	24.2
5	Second assembly (overall level)	0.6	0.6	0.2
6	Matrix factorization	33.0	33.0	11.8
7	Solution (fwd. reduce, bk. substitute)	2.6	2.6	0.9
8	Surface stress recovery	5.5	5.5	5.5
	All subsequent response analysis			
1A	Accounting	6.0	——	——
2A	Numerical integration	39.0	7.8	7.8
3A	First assembly (zone level)	14.4	4.4	4.4
4A	Condensation	——	——	——
5A	Second assembly (overall level)	0.6	0.2	0.2
6A	Matrix factorization	33.0	33.0	11.8
7A	Solution (fwd. reduce, bk. substitute)	2.6	2.6	0.9
8A	Surface stress recovery	5.5	5.5	5.5
	Design sensitivity analysis (one design variable)			
9	Accounting	6.0	——	——
10	Numerical integration	39.0	4.8	4.8
11	First assembly (zone level)	14.4	4.4	4.4
12	Condensation	——	——	——
13	Second assembly (overall level)	0.6	0.5	0.5
13A	Matrix factorization	33.0	——	——
14	Solution (fwd. reduce, bk. substitute)	2.6	7.5	2.6
15	Surface stress sensitivity recovery	5.5	4.6	4.6
	Shape optimization (36 analyses, 3 DSA's with 7 design variables)			
16	First response analysis	101.1	101.1	102.0
17	35 Subsequent response analyses	35 × 101.1 = 3538.5	35 × 53.5 = 1872.5	35 × 30.6 = 1071.0
18	3 Design sensitivity analyses	21 × 101.1 = 2123.1	21 × 21.8 = 457.8	21 × 16.9 = 354.9
19	Total shape-optimization time	5762.7	2431.4	1527.9

[a]UP/FD: univariate perturbation/forward difference.

Table 3 Test cases for the thin circular disk under external pressure

Case	Zone I Condense	Zone I Expand	Zone II Condense	Zone II Expand	Geometry sensitivity
I	No	No	No	No	Full
II	No	No	No	No	Partial
III	Yes	No	Yes	No	Partial
IV	Yes	Yes	Yes	No	Partial
V	Yes	No	Yes	Yes	Partial
VI	Yes	Yes	Yes	Yes	Partial

Table 4 CPU timings for substructured design sensitivity analyses of a thin circular disk under external pressure

Item no.	Step description	Analysis cases I	II	III	IV	V	VI
		Response analysis					
1	Accounting	0.7	0.7	0.6	0.6	0.6	0.6
2	Integration	11.5	11.7	11.7	11.7	11.6	11.7
3	First assembly	1.1	1.0	1.0	1.2	1.4	1.5
4	Condensation	——	——	1.7	1.9	1.9	2.0
5	Second assembly	——	——	——	——	——	——
6	Matrix factorization	1.4	1.4	0.1	0.1	0.1	0.1
7	Solution and expansion	0.4	0.4	0.1	0.3	0.5	0.7
8	Stress recovery	0.9	0.9	0.8	0.8	0.8	0.9
		Design sensitivity analysis					
9	Accounting	——	——	——	——	——	——
10	Integration	42.7	13.9	14.0	14.0	13.9	14.0
11	First assembly	1.2	1.0	0.8	0.6	0.6	0.5
12	Condensation	——	——	1.0	1.0	1.1	1.1
13	Second assembly	0.1	0.1	——	——	——	——
14	Solution and expansion	0.8	0.8	0.2	0.3	0.4	0.6
15	Stress sensitivity recovery	0.7	0.8	0.4	0.5	0.6	0.8
16	Total	61.5	32.7	32.4	33.0	33.6	34.5

Partial radial sensitivity:

$$R = \begin{cases} a + 2t(b/2 - a) & 0 \le t \le 0.5 \\ bt & 0.5 \le t \le 1 \end{cases}$$

$$R_{,a} = \begin{cases} 1 - 2t & 0 \le t \le 0.5 \\ 0 & 0.5 \le t \le 1 \end{cases}$$

where a and b are the inner and outer radii, respectively. The inner radius a has been taken as the design variable. Six different cases for these examples were studied and are shown in Table 3. For the case with full geometric sensitivity, no condensation was performed. For the case with partial geometric sensitivity, different combinations of condensation and expansion of zones I and II were considered. The CPU timings for all of these studies were reported in Table 4. In comparing the timings for case I with the rest of the cases shown in this table, it is clear that substructuring is a powerful tool for gaining economy when partial geometric sensitivity is present in the design. The other advantages observed in the previous example, namely, the economy in the substructured matrix factorization step, the economy in the substructured solution and expansion step, the offset of additional timings due to condensation by savings in solution and expansion step for multiple calculations, are all evident in this example also from the timings shown in Table 4. As in the previous examples, the sensitivities computed in this example problem by both the substructured and unsubstructured models were in agreement up to five significant figures.

Conclusions

The concept of exact condensation for the design sensitivity analysis of large structures with partial geometric sensitivity to design variables is presented using the multizone boundary-element method. This concept allows the overall system of equations to be reduced to retain only that part of the model which is allowed to change and to be iterated upon during the numerical shape optimization process. Starting from the reduced boundary-element equations, the reduced design sensitivity analysis equations are derived through implicit differentiation of the former set of equations. The equations for expansion to obtain sensitivities of the condensed portions are similarly derived. An exact formulation is described which preserves the accuracy of the results for the condensed sensitivity equations. The present development allows the boundary-element zones to be arbitrarily selected for condensation. It also allows for the simultaneous existence of both multiple condensed and multiple uncondensed zones in the reduced model.

The condensation and the subsequent expansion equations for reduced sensitivity analysis require the apparent computation of sensitivities of submatrix (partition) inverses. An approach to eliminate the need for such explicit computations is presented. The method is applied for the sensitivity analysis and optimization of two-dimensional plane and axisymmetric cases. CPU timings for both full and the reduced sensitivity analysis are presented for comparison. The reduced analysis involves a slight initial additional resource expenditure for the setup, accounting, storage, and assembly requirements of the condensed matrices. However, substantial economies are seen for all of the subsequent matrix factorizations required in shape optimization. Since the solution step is repeated numerous times in the numerical optimization process, the savings accrued in this step during reduced sensitivity analysis are important. For the overall process, these savings in each of the solution steps accumulate to provide substantial economy in overall CPU times. The present development constitutes a significant step towards enhancing the efficiency of boundary-element design sensitivity analyses and facilitating its use on large-scale shape-optimization problems of practical utility.

Acknowledgment

The research reported in this paper was supported by the National Science Foundation Grant DMC-8708254 with Tony Woo as the grant monitor. The authors also wish to acknowledge the help of Anil Gupta and B. L. Keshavakumar in debugging the computer program.

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